JOURNAL OF APPROXIMATION THEORY 7, 143–154 (1973)

On Uniform Approximation by Certain Generalized Spline Functions*

JOSEPH W. JEROME

Department of Mathematics, Northwestern University, Evanston, Illinois 60201

Communicated by Oved Shisha

Received September 6, 1970

INTRODUCTION

If $f \in C^{k-1}[a, b]$, $k > 1$, then deBoor has shown [7] that, for a given partition Δ of $[a, b]$, one can construct polynomial spline functions s, of degree $k - 1$, with knots of prescribed multiplicity in Δ, satisfying, for $0 \leqslant j \leqslant k - 1$,

$$\sup_{x\in[a,b]} | D^j f(x) - D^j s(x)| \leqslant c(\bar{\Delta})^{k-j-1}\, \omega(D^{k-1}f; \bar{\Delta}) \tag{1}$$

where c is independent of Δ and f, ω denotes the modulus of continuity of $D^{k-1}f$, and $\bar{\Delta}$ denotes the maximum mesh spacing. If $D^{k-1}f$ is of bounded variation on $[a, b]$, then (1) implies the obvious inequality

$$\sup_{x\in[a,b]} | D^j f(x) - D^j s(x)| \leqslant c(\bar{\Delta})^{k-j-1}\, V_{\bar{\Delta}}(D^{k-1}f) \tag{2}$$

where $V_{\bar{\Delta}}(g)$ denotes the supremum of the variations in g over intervals of length not exceeding $\bar{\Delta}$. If, moreover, $D^k f$ exists and is bounded on $[a, b]$, then (2) implies

$$\sup_{x\in[a,b]} | D^j f(x) - D^j s(x)| \leqslant c(\bar{\Delta})^{k-j} \sup_{x\in[a,b]} | D^k f(x)| \,. \tag{3}$$

Much of the activity in the investigation of error estimation in uniform approximation by spline functions has centered about obtaining estimates of the form (3), frequently under the additional assumption that $D^k f \in C[a, b]$. DeBoor himself obtained the result (3), for $j = 0$, somewhat earlier [6], using local spline approximation by moments. DeBoor's work in [6] was a refinement of earlier work of Birkhoff [2] who had considered the cases $k = 2m$ and had made mesh restrictions not assumed either in [6] or in [7].

* This research was supported in part by NSF Grant GP-19526.

In the case of uniform approximation by interpolating polynomial spline functions, estimates of the form (1) for uniform meshes in the cases $k = 2m$ have been obtained by Swartz and Varga [14], extending earlier results of the first author [13]. Although simple interpolation conditions are imposed at interior mesh points, full sets of endpoint constraints appear necessary, as described in [14]. Earlier, Birkhoff and deBoor [3] had obtained estimates of the form (3), using almost uniform meshes, in the cases $k = 2$ and $k = 4$, and Ahlberg, Nilson, and Walsh [1] had established such results for $k = 2m$ with f and s periodic and with uniform mesh spacing.

Similar results for uniform approximation by interpolating polynomial spline functions are available when the splines are so-called Hermite splines of odd degree $2m - 1$ between knots, with maximum contact through derivatives of order $m - 1$ at each knot. In this case, (3) holds for $k = 2m$ with no restriction whatsoever on Δ (see Birkhoff, Schultz, and Varga [4] where a stronger version of this theorem is proved).

In the case when polynomial interpolating spline functions are replaced by interpolating spline functions locally annihilated by operators of the form L^*L, L a nonsingular linear differential operator of order m, estimates of the form (3), or estimates with $\sup_{x\in[a,b]} | D^{2m}f(x)|$ replaced by $\sup_{x\in[a,b]} | L^*Lf(x)|$ in (3), are available only in the case of full contact, i.e., Hermite, interpolation at the knots (see [14]). Otherwise, one can establish only that the exponent of $\bar{\Delta}$ is $2m - j - \frac{1}{2}$ and, if $m \leqslant j \leqslant 2m - 1$, a mesh restriction is required (see, e.g., Schultz and Varga [12]). This estimate, however, is valid for any f such that $D^{2m}f \in L_2[a, b]$.

It is the purpose of this paper to construct spline functions $s \in C^{k-2}[a, b]$, locally annihilated (between preassigned knots) by nonsingular linear differential operators Λ of order k, satisfying estimates of the form (2), which are of course stronger than the estimates of the form (3). We do not assume differentiability of $D^{k-1}f$; it need only be of bounded variation. Mesh restrictions of the form $\sup \Delta / \inf \Delta \leqslant \alpha < \infty$ are imposed, however. Here, Λ need not be self-adjoint, but may be an arbitrary nonsingular linear differential operator with suitable coefficients on a closed interval $[a, b]$; self-adjoint operators with continuous coefficients are included. Λ is thus a natural generalization of D^k and as such possesses minimal support splines, which extend the basic polynomial splines of degree $k - 1$, and which determine integral kernels, in terms of which the approximation error may be conveniently expressed.

The technique employed which enables us to obtain the stronger (2) is that of local factorizations of Λ of the form $\Lambda = \varphi D \Omega$ for some nonvanishing function φ and nonsingular operator Ω of order $k - 1$. It is possible to decompose $[a, b]$ into the finite union of adjacent closed intervals on which Λ may be so represented.

I wish to express my gratitude to Professor Michael Golomb, who suggested to me the technique of local factorization, and who also suggested the method of proof in Lemma 1 below.

1. Convergence Results

Let Λ be a linear differential operator on $[a, b]$ with real coefficients given by

$$\Lambda = D^k + \sum_{j=0}^{k-1} a_j D^j, \qquad k \geqslant 1, \tag{1.1}$$

where $a_j \in C^j[a, b]$, $0 \leqslant j < k$. Let η_Λ denote the null space of Λ. Define $\theta(\cdot, \xi) \in \eta_\Lambda$ for each $\xi \in [a, b]$ by:

$$[D_x{}^j \theta(x, \xi)]_{x=\xi} = \delta_{j,k-1}, \qquad 0 \leqslant j \leqslant k-1. \tag{1.2}$$

It is known (see e.g., [16, pp. 75–78]) that the function θ has the representation, on $[a, b] \times [a, b]$,

$$\theta(x, \xi) = \sum_{j=1}^{k} u_j(x)\, u_j{}^*(\xi) \tag{1.3}$$

where $\{u_j\}_1{}^k$ are a basis for η_Λ and where $\{u_j{}^*\}_1{}^k$ are the functions in the last column of $W^{-1}[u_1, \ldots, u_k]$, where the Wronskian $W[u_1, \ldots, u_k]$ is given by $(W_{ij}) = (u_j^{(i-1)})$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant k$. The functions $\{u_j{}^*\}_1^k$ are in the null space of the adjoint operator Λ^* given by

$$\Lambda^* f = (-1)^k D^k f + \sum_{j=0}^{k-1} (-1)^j D^j(a_j f). \tag{1.4}$$

Λ^* is a nonsingular linear differential operator of order k with basis $\{u_j{}^*\}_1^k$.

If $a = x_0 < x_1 < \cdots < x_n = b$ is a partition of $[a, b]$ the function $s \in C^{k-2}[a, b]$ is called a Λ-spline with simple knots at the points $\{x_i\}$ if $s \in C^k(\bigcup_i (x_i, x_{i+1}))$ and $\Lambda s(x) = 0$ if $x \neq x_i$, $0 \leqslant i \leqslant n$. The numbers $\bar{\Delta} = \max_i (x_{i+1} - x_i)$ and $\underline{\Delta} = \min_i (x_{i+1} - x_i)$ are the maximum and minimum mesh spacings. Notice that the function $\hat{\theta}(x, \xi)$ defined by

$$\hat{\theta}(x, \xi) = \begin{cases} \theta(x, \xi) & \text{if } x \geqslant \xi \\ 0 & \text{if } x < \xi \end{cases} \tag{1.5}$$

is a Λ-spline with the single interior knot ξ. For example if $\Lambda = D^k$ then

$$\hat{\theta}(x, \xi) = (x - \xi)_+^{k-1}/(k-1)!\,.$$

The operators $\Lambda = D^k$ and $\Lambda = D(D^2 + 1^2) \cdots (D^2 + m^2)$ have the property of global factorization in the form $D\Omega$, Ω nonsingular of order one less than Λ. More generally, if Λ is any nonsingular operator, it admits the local factorization $\varphi D\Omega$ mentioned earlier. This is of central importance in the error estimation and will be described now, following Golomb [9] and Karlin and Studden [11, Ch. 8, §5].

If x_* is any point of $[a, b]$, let the functions $\psi_i \in \eta_\Lambda$, $0 \leqslant i \leqslant k-1$, be defined by

$$D^j\psi_i(x_*) = \delta_{ij}, \qquad 0 \leqslant j \leqslant k-1. \tag{1.6}$$

The Wronskians $W[\psi_0]$, $W[\psi_0, \psi_1], \ldots, W[\psi_0, \psi_1, \ldots, \psi_{k-1}]$ have the value 1 at x_*, hence are greater than or equal to some fixed positive constant in some closed interval I_* containing x_* and contained in $[a, b]$. We agree to call such an interval a Pólya interval for the operator Λ. Defining on I_* the functions,

$$\begin{aligned} &w_0 = \psi_0,\ w_1 = W[\psi_0, \psi_1]/\psi_0^2, \ldots, \qquad \text{and, for } 2 \leqslant i \leqslant k-1 \\ &w_i = W[\psi_0, \ldots, \psi_i]\, W[\psi_0, \ldots, \psi_{i-2}]/W^2[\psi_0, \ldots, \psi_{i-1}], \end{aligned} \tag{1.7}$$

it is clear that $w_0, w_1, \ldots, w_{k-1}$ are strictly positive functions with w_i in $C^{k-i}(I_*)$. Define, for $u \in C^1(I_*)$,

$$D_i u = D(u/w_i). \tag{1.8}$$

Then we have the factorization on I_*,

$$\Lambda = w_0 w_1 \cdots w_{k-1} D_{k-1} D_{k-2} \cdots D_0 . \tag{1.9}$$

In particular Λ has the form

$$\Lambda = \varphi D\Omega, \tag{1.10}$$

where $\varphi = w_0 \cdots w_{k-1}$ and $\Omega = (1/w_{k-1})\, D_{k-2} \cdots D_0$. It is clear that there exists a finite number of Pólya intervals, $I_1, I_2, \ldots, I_q$ such that $I_\nu \cap I_\mu$ has at most one point if $\nu \neq \mu$ and such that $\bigcup_\nu I_\nu = [a, b]$. By (1.10), on each set I_ν, $1 \leqslant \nu \leqslant q$, Λ has the factorization

$$\Lambda = \varphi_\nu D\Omega_\nu \tag{1.11}$$

where $\varphi_\nu \in C^1[I_\nu]$ and does not vanish on I_ν and where Ω_ν is a nonsingular operator of order $k-1$.

We are now prepared to state the major theorem of the paper. We write $\sup \Delta$ for $\bar{\Delta}$ and $\inf \Delta$ for $\underline{\Delta}$.

THEOREM 1. *If Λ is an arbitrary nonsingular linear differential operator satisfying* (1.1), *if $k \geqslant 1$ and if $f \in C^{k-1}[a, b]$ with $D^{k-1}f$ of bounded variation on $[a, b]$ and if $\mathscr{D}_\alpha$ is a collection of partitions of $[a, b]$ with* $\sup \Delta / \inf \Delta \leqslant \alpha$

for all $\Delta \in \mathscr{D}_\alpha$, then for each Δ in $\mathscr{D}_\alpha$ such that $\bar{\Delta}$ is sufficiently small and such that Δ contains at least $2k + 1$ points there exists a Λ-spline s with knots in Δ satisfying, for $0 \leqslant j \leqslant k - 1$,

$$\sup_{x \in [a,b]} | D^j f(x) - D^j s(x)| \leqslant c(\bar{\Delta})^{k-j-1} \max_{1 \leqslant \nu \leqslant q} V_{\bar{\Delta}}(\Omega_\nu f) \tag{1.12}$$

where the Ω_ν are the operators of (1.11) *of order $k - 1$. Here, $V_{\bar{\Delta}}(g)$ represents the supremum of the variations of g over subintervals of $[a, b]$ of length not exceeding $\bar{\Delta}$ and c is independent of f and $\Delta \in \mathscr{D}_\alpha$. In particular,* (2) *holds and, if $D^k f$ exists and is bounded on* $[a, b]$, (1.12) *also implies,*

$$\sup_{x \in [a,b]} | D^j f(x) - D^j s(x)| \leqslant c'(\bar{\Delta})^{k-j} \sup_{y \in [a,b]} | \Lambda f(y)| . \tag{1.13}$$

Remark. If $j = k - 1$, the supremum in (1.12) and (1.13) is understood to be taken over $x \notin \Delta$, since in this case $D^{k-1}s$ does not exist at the points of Δ. In what follows, if ζ is a Λ-spline, $D^{k-1}\zeta$ is undefined at the knots of ζ.

We shall defer the proof of Theorem 1 to Section 3, after we obtain, in Section 2, some preliminary results of independent interest. In particular, we shall construct generalizations of the well-known basic splines of compact support. These will play a basic role in Section 3, in the proof of Theorem 1. As an interesting application of the ideas, we explicitly compute, at the end of Section 3, the generalized basic trigonometric splines of degree m, i.e., the case $\Lambda = D(D^2 + 1^2) \cdots (D^2 + m^2)$. We now close Section 1 with an additional remark and corollary.

Remark. It is possible to weaken the smoothness assumptions on the coefficients a_j of Λ. In fact, if Λ is any operator of the form (1.1) with continuous coefficients on $[a, b]$ such that Λ has an adjoint operator

$$\Lambda^* = (-1)^k D^k + \sum_{j=0}^{k-1} b_j D^j, \qquad k \geqslant 1,$$

with continuous coefficients on $[a, b]$, then Theorem 1 holds. In particular, if Λ is of the form (1.1), is self-adjoint and has continuous coefficients then Theorem 1 holds. We thus obtain the following corollary, with $\Lambda = L^*L$.

COROLLARY 1. *If $L = \sum_{j=0}^m a_j D^j$ with $a_m(x) \equiv 1$, $m \geqslant 1$,* on $[a, b]$ and *$a_j \in C^j[a, b]$ and if Δ is a sufficiently fine partiton of $[a, b]$ containing at least $4m + 1$ points then the estimate, for $0 \leqslant j \leqslant 2m - 1$,*

$$\sup_{x \in [a,b]} | D^j f(x) - D^j s(x)| \leqslant c(\bar{\Delta})^{2m-j-1} V_{\bar{\Delta}}(D^{2m-1} f) \tag{1.14}$$

holds, where $f \in C^{2m-1}[a, b]$, $D^{2m-1} f$ *is of bounded variation on* $[a, b]$, s *is an* L^*L*-spline (frequently called an L-spline) with knots in* Δ *and* c *is independent of* f *and of* Δ *as before. If* $D^{2m}f$ *exists and is bounded, then the right side of this latter expression may be replaced by either a constant factor times* $(\bar{\Delta})^{2m-j} \sup_{x\in[a,b]} | D^{2m}f(x)|$ *or times* $(\bar{\Delta})^{2m-j} \sup_{x\in[a,b]} | L^*Lf(x)|$, *respectively.*

2. Generalized B-Splines

Let $a \leqslant t_0 < t_1 < \cdots < t_k \leqslant b$ be any set of $k + 1$ consecutive points in $[a, b]$. If $\{u_i\}$ is a fixed basis of η_Λ, and $\{u_j^*\}$ are the corresponding adjoint functions, i.e., $u_i^*,\ldots, u_k^*$ are the elements of the last column of $W^{-1}[u_1,\ldots, u_k]$, or fixed constant multiples thereof, let $\beta_0,\ldots, \beta_k$ be any solution of the system

$$\begin{aligned} \beta_0 u_1^*(t_0) + \beta_1 u_1^*(t_1) + \cdots + \beta_k u_1^*(t_k) &= 0 \\ \beta_0 u_2^*(t_0) + \beta_1 u_2^*(t_1) + \cdots + \beta_k u_2^*(t_k) &= 0 \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad & \\ \beta_0 u_k^*(t_0) + \beta_1 u_k^*(t_1) + \cdots + \beta_k u_k^*(t_k) &= 0. \end{aligned} \tag{2.1}$$

Then we have the following theorem.

Theorem 2. *The function*

$$M(x) = M(x; t_0,\ldots, t_k) = \sum_{j=0}^{k} \beta_j \hat{\theta}(x, t_j)$$

is a Λ*-spline with knots at* $t_0,\ldots, t_k$ *whose support is contained in* $[t_0, t_k]$.

Proof. We need only show that $x \geqslant t_k \Rightarrow M(x) = 0$. But if $x \geqslant t_k$, then by (1.3),

$$M(x) = \sum_{j=0}^{k} \beta_j \theta(x, t_j) = \sum_{j=0}^{k} \beta_j \left\{ \sum_{\mu=1}^{k} u_\mu(x)\, u_\mu^*(t_j) \right\},$$

where $\theta(x, t) = \sum_{\mu=1}^{k} u_\mu(x)\, u_\mu^*(t)$, which is seen to be zero upon multiplication of the μth equation of (2.1) by $u_\mu(x)$ and the addition of the resultant columns. This completes the proof of Theorem 2.

Nontrivial solutions of (2.1) always exist. Furthermore, if $t_k - t_0$ is sufficiently small it follows from theorems on the zeroes of null solutions of nonsingular linear differential operators (see [17, p. 346]) that any nontrivial solution possesses the property that every $\beta_j \neq 0, 0 \leqslant j \leqslant k$. If $\Lambda^* = (-1)^k D^k$, the functions $u_\mu^*(t)$ may be chosen to be the functions $t^{\mu-1}$, $1 \leqslant \mu \leqslant k$, and the numbers β_j may be chosen to satisfy

$$\beta_j = k! \Big/ \prod_{i\neq j} (t_i - t_j), \qquad 0 \leqslant j \leqslant k.$$

In this case, M is simply the well-known fundamental spline (see Curry and Schoenberg [5]).

$$M(x) = \sum_{j=0}^{k} \frac{k}{\prod_{i\neq j}(t_i - t_j)} (x - t_j)_+^{k-1}. \tag{2.2}$$

A particularly simple expression is available for the coefficients β_j if we introduce the functions

$$V_j(x) = \begin{vmatrix} u_1^*(t_1) & \cdots & u_1^*(t_{j-1}) & u_1^*(t_{j+1}) & \cdots & u_1^*(t_k) & u_1^*(x) \\ & & & \cdots\cdots & & & \\ u_k^*(t_1) & \cdots & u_k^*(t_{j-1}) & u_k^*(t_{j+1}) & \cdots & u_k^*(t_k) & u_k^*(x) \end{vmatrix} \tag{2.3}$$

where $1 \leqslant j \leqslant k$. Then, clearly,

$$\beta_j V_j(t_j) = -\beta_0 V_j(t_0), \qquad 1 \leqslant j \leqslant k. \tag{2.4}$$

For the estimation of the error in Section 3, it will be necessary to show that $|\beta_j/\beta_0|$ are bounded, $1 \leqslant j \leqslant k$, independently of Δ, provided $\sup \Delta/\inf \Delta \leqslant \alpha$. We will also show that $|\beta_j/\beta_0|$ is a differentiable function of t_0. Specifically, we have the following lemma.

LEMMA 1. *Let* $\bar{\Delta} = \max_i(t_{i+1} - t_i)$ *and* $\inf \Delta = \min_i(t_{i+1} - t_i)$ *with* $\bar{\Delta}$ *sufficiently small. Let* $\beta_0, \ldots, \beta_k$ *solve* (2.1) *with* $\beta_0 \neq 0$ *and suppose* $\sup \Delta/\inf \Delta \leqslant \alpha$ *for some positive constant* α. *Then* β_j/β_0 *is a differentiable function of* t_0 *and there is a constant* C, *independent of the choice of* $\{t_i\}$, *such that*

$$|\beta_j/\beta_0| \leqslant C, \qquad 1 \leqslant j \leqslant k. \tag{2.5}$$

Proof. To prove (2.5) we observe, by (2.4), that

$$\beta_j/\beta_0 = -V_j(t_0)/V_j(t_j). \tag{2.6}$$

Using only elementary properties of determinants, in this case, subtracting the first column of the respective determinants of $V_j(t_0)$ and $V_j(t_j)$ from columns 2 through k and dividing by appropriate factors, then subtracting column 2 from 3 through k, etc., we obtain (2.7) below where i has values between 1 and k

$$\frac{V_j(t_0)}{V_j(t_j)} = \frac{\prod_{i\neq j}(t_0 - t_i)}{\prod_{i\neq j}(t_j - t_i)} \times \frac{\begin{vmatrix} u_1^*(t_1) & u_1^*(t_1, t_2) & \cdots & u_1^*(t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_k, t_0) \\ \vdots & \vdots & & \vdots \\ u_k^*(t_1) & u_k^*(t_1, t_2) & \cdots & u_k^*(t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_k, t_0) \end{vmatrix}}{\begin{vmatrix} u_1^*(t_1) & u_1^*(t_1, t_2) & \cdots & u_1^*(t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_k, t_j) \\ \vdots & \vdots & & \vdots \\ u_k^*(t_1) & u_k^*(t_1, t_2) & \cdots & u_k^*(t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_k, t_j) \end{vmatrix}} \tag{2.7}$$

where the ν-th column of each determinant consists of $(\nu - 1)$ order divided differences of the basis functions taken on the indicated points. Using the representation for ν-th order divided differences,

$$f(\xi_1, ..., \xi_{\nu+1}) = \frac{1}{\nu!} \int_{-\infty}^{\infty} M(x) f^{(\nu)}(x)\, dx, \tag{2.8}$$

where M is given by (2.2) with ν in place of k and ξ_{i+1} in place of t_i, we obtain the estimate

$$| f(\xi_1, ..., \xi_{\nu+1}) - f^{(\nu)}(\xi_1)/\nu! \,| \leqslant \frac{1}{\nu!}\, \omega(D^\nu f; \max_i \xi_i - \min_i \xi_i) \tag{2.9}$$

where the points $\xi_1, ..., \xi_{\nu+1}$ need not be in order. (2.9) follows from (2.8) and the property

$$\int_{-\infty}^{\infty} M(x)\, dx = 1.$$

Now if $\gamma > 0$ is such that the Wronskian satisfies

$$| W[u_1^*(x), ..., u_k^*(x)]| \geqslant \gamma > 0, \qquad x \in [a, b]$$

it is clear upon using (2.9) in (2.7) with $\xi_1 = t_1$ and $\nu = 0, 1, ..., k - 1$, that we can choose $\bar{\Delta}$ sufficiently small so that

$$| V_j(t_0)/V_j(t_j)| \leqslant 2 \,| W[u_1^*(t_1), ..., u_k^*(t_1)]| (\alpha k)^{k-1}/\gamma$$

which establishes (2.5). The differentiability of β_j/β_0 as a function of t_0 follows immediately from (2.3) and (2.6).

3. Spline Constructions

This section will be devoted to the proof of Theorem 1 and, in particular, to the construction of the noninterpolating Λ-spline approximation s to the given function f.

Proof of Theorem 1. If $f \in C^{k-1}[a, b]$, and $D^{k-1}f$ is of bounded variation, then f has the Stieltjes integral representation, for each $x \in [a, b]$,

$$f(x) = u(x) + \sum_{\nu=1}^{q} \int_{I_\nu} \hat{\theta}(x, \xi)\, \varphi_\nu(\xi)\, d(\Omega_\nu f)(\xi) \tag{3.1}$$

where $u \in \eta_\Lambda$ is defined by $D^j u(a) = D^j f(a)$, $0 \leqslant j \leqslant k-1$. By assumption, Δ has at least $2k+1$ points. For each $\xi \in [a, b]$ let the points $t_1(\xi), \ldots, t_k(\xi)$ be consecutive points of Δ to the right of ξ (left, if necessary) satisfying

$$\begin{gathered} \underline{\Delta} \leqslant | t_1 - \xi | \leqslant 2\bar{\Delta} \\ | t_1 - \xi | < | t_2 - \xi | < \cdots < | t_k - \xi | . \end{gathered} \tag{3.2}$$

Notice that we can choose $t_1(\xi), \ldots, t_k(\xi)$ invariantly with respect to ξ for ξ between any two successive knots of Δ, say for $\xi \in [x_\nu, x_{\nu+1})$. Now define, for each $\xi \in [a, b]$, the function $\gamma(\cdot, \xi)$ as follows. We make the convention that $t_0 = \xi$.

$$\gamma(x, \xi) = \hat{\theta}(x, \xi) - (1/\beta_0) \sum_{j=0}^{k} \beta_j \hat{\theta}(x, t_j). \tag{3.3}$$

Here the points $t_1, \ldots, t_k$ are chosen for each ξ as just indicated and the numbers $\beta_0, \ldots, \beta_k$ are a normalized nontrivial solution of (2.1). Notice that $\beta_0, \ldots, \beta_k$ are differentiable in ξ for $\xi \in (x_\nu, x_{\nu+1})$. It is clear that $\gamma(\cdot, \xi)$ is a Λ-spline with knots in Δ for each fixed ξ. Notice that ξ is not a knot of $\gamma(\cdot, \xi)$. Also, $\gamma(\cdot, \xi) - \hat{\theta}(\cdot, \xi)$ is a Λ-spline with knots at $\xi, t_1, \ldots, t_k$ and has support confined to the closed interval with endpoints ξ and t_k. This in turn implies that, for each fixed x, $\gamma(x, \cdot) - \hat{\theta}(x, \cdot)$ has support on the interval

$$J_x = [a, b] \cap [x - (k+1)\bar{\Delta}, x + (k+1)\bar{\Delta}]. \tag{3.4}$$

In the case that Λ is taken to be D^k, the function $\gamma(\cdot, \xi) - \hat{\theta}(\cdot, \xi)$ is a basic polynomial spline. Similar constructions for this case have been considered by Dailey [8] and Ziegler [15] in the study of L_1 approximation.

We are now prepared to define s. For each $x \in [a, b]$ we define

$$s(x) = u(x) + \sum_{\nu=1}^{q} \int_{I_\nu} \varphi_\nu \gamma(x, \xi)\, d(\Omega_\nu f)(\xi) \tag{3.5}$$

where $u \in \eta_\Lambda$ is defined by (3.1). It is clear that (3.5) defines a Λ-spline with knots in Δ. We propose to investigate $| D^j f(x) - D^j s(x)|$ for $0 \leqslant j \leqslant k-1$. For x fixed in $[a, b]$ we have, by (3.1) and (3.5),

$$\begin{aligned} | D^j f(x) - D^j s(x)| &= \left| \sum_{\nu=1}^{q} \int_{I_\nu} \varphi_\nu D_x^{\,j}\{\hat{\theta}(x, \xi) - \gamma(x, \xi)\}\, d(\Omega_\nu f)(\xi) \right| \\ &= \left| \sum_{\nu=1}^{q} \int_{I_\nu \cap J_x} \varphi_\nu D_x^{\,j}\{\hat{\theta}(x, \xi) - \gamma(x, \xi)\}\, d(\Omega_\nu f)(\xi) \right| . \end{aligned} \tag{3.6}$$

Designating the subintervals of $I_\nu \cap J_x$ determined by the points of Δ by $I_{\nu\mu}$ and using (3.3) in (3.6) with $\xi = t_0$ we have

$$| D^j f(x) - D^j s(x)| = \left| \sum_\nu \sum_\mu \int_{I_{\nu\mu}} \varphi_\nu D_x{}^j \left\{ \sum_{i=0}^{k} (\beta_i/\beta_0)\, \hat{\theta}(x, t_i) \right\} d(\Omega_\nu f)(t_0) \right| . \tag{3.7}$$

Now using the Taylor expansion

$$\theta(x, y) = \frac{(x-y)^{k-1}}{(k-1)!} + \int_y^x \frac{(x-t)^{k-1}}{(k-1)!} D_t{}^k \theta(t, y)\, dt \tag{3.8}$$

(see (1.2)) we obtain

$$D_x{}^j \hat{\theta}(x, y) = ((x-y)_+^{k-j-1}/(k-j-1)!) + O((x-y)_+^{k-j}) \tag{3.9}$$

where the constant in the order expression does not depend on x or y. Upon substituting (3.9) into (3.7) and making use of Lemma 1, we obtain

$$| D^j f(x) - D^j s(x)| \leqslant \tilde{c} \sum_\nu \sum_\mu \int_{I_{\nu\mu}} | \varphi_\nu(t_0)| \sum_{i=0}^{k} (x - t_i)_+^{k-j-1} \,| d(\Omega_\nu f)(t_0)| \tag{3.10}$$

and the final result of (1.12) is immediate from (3.10), since the number of integrals on the right side is at most $[2(k+1)\,\alpha + 2q]$, where the bracket denotes the greatest integer, less than or equal to the quantity indicated.

We shall close the paper by computing the kernel $\theta(x, \xi)$ corresponding to the operator $\Lambda = D(D^2 + 1^2) \cdots (D^2 + m^2)$ and suitable coefficients β_ν. In this case, 1, $\sin x$, $\cos c$,..., $\sin mx$, $\cos mx$ for a basis form the $(2m+1)$ dimensional null space of Λ. It is readily seen that the function

$$\theta(x, \xi) = \sum_{\nu=0}^{m} \alpha_\nu \cos \nu(x - \xi), \tag{3.11}$$

where the coefficients α_ν solve the system,

$$\sum_{\nu=0}^{m} \nu^{2p}(-1)^p\, \alpha_\nu = \delta_{p,m}\,, \qquad 0 \leqslant p \leqslant m, \tag{3.12}$$

satisfies (1.2) with $k = 2m + 1$. In (3.12) we have adopted the convention that $0^0 = 1$. Using Cramer's rule and the expansion for the Vandermonde determinant, we readily compute

$$\alpha_0 = \frac{1}{(m!)^2}\,, \qquad \alpha_\nu = 1 \Big/ \prod_{\substack{j=0\\ j\neq\nu}}^{m} (j^2 - \nu^2), \qquad 1 \leqslant \nu \leqslant m. \tag{3.13}$$

If the basis, $\{e^{ijx}\}_{j=-m}^{j=m}$, is chosen in (2.1) the coefficients β_ν may be readily obtained. They are given by

$$\beta_\nu = \frac{e^{imt_\nu}}{\prod_{\substack{j=0\\ j\neq\nu}}^{2m+1} (e^{it_j} - e^{it_\nu})}, \qquad 0 \leqslant \nu \leqslant 2m+1. \tag{3.14}$$

It is clear that the function

$$M(x; t_0, \ldots, t_{2m+1}) = \sum_{\nu=0}^{2m+1} \mathrm{Re} \left\{ \frac{e^{imt_\nu}}{\prod_{\substack{j=0\\ j\neq\nu}}^{2m+1} (e^{it_j} - e^{it_\nu})} \right\} \left[\sum_{\mu=0}^{m} \alpha_\mu \cos \mu(x - t_\nu) \right]_+$$

is a trigonometric B-spline of order m with knots at $t_0, \ldots, t_{2m+1}$ and support on the interval $[t_0, t_{2m+1}]$.

References

1. J. H. Ahlberg, E. N. Nilson, and J. L. Walsh, "The Theory of Splines and Their Applications," Academic Press, New York (1967).
2. Garrett Birkhoff, Local spline approximation by moments, *J. Math. Mech.* **16** (1967), 987–990.
3. Garrett Birkhoff and Carl deBoor, Error bounds for spline interpolation, *J. Math. Mech.* **13** (1964), 727–735.
4. Garrett Birkhoff, M. H. Schultz, and R. S. Varga, Piecewise Hermite interpolation in one and two variables with applications to partial differential equations, *Numer. Math.* **11** (1968), 232–256.
5. H. B. Curry and I. J. Schoenberg, On Pólya Frequency Functions IV: The fundamental spline functions and their limits, *J. Analyse Math.* **17** (1966), 71–107.
6. Carl deBoor, A note on local spline approximation by moments, *J. Math. Mech.* **17** (1968), 729–736.
7. Carl deBoor, On uniform approximation by splines, *J. Approximation Theory* **1** (1968), 219–235.
8. J. W. Dailey, Approximation by spline-type functions and related problems, dissertation, Case Western Reserve University, September, 1969.
9. Michael Golomb, Rates of Convergence of L-splines which interpolate H_L-functions, MRC Technical Summary Report 1048, Madison, Wisconsin, 1970.
10. T. N. E. Greville, Interpolation by generalized spline functions, MRC Technical Summary Report 476, Madison, Wisconsin, 1964.
11. S. J. Karlin and W. J. Studden, "Tchebycheff Systems in Analysis and Statistics," Wiley, New York, 1966.
12. M. H. Schultz and R. S. Varga, L-splines, *Numer. Math.* **10** (1967), 345–369.
13. Blair Swartz, $o(h^{2n+2-l})$ bounds on some spline interpolation errors, BAMS **74** (1968), 1072–1078.
14. Blair K. Swartz and Richard S. Varga, Error bounds for spline and L-spline interpolation, *J. Approximation Theory* **6** (1972), 6–49.

15. ZVI ZIEGLER, One-sided L_1-approximation by splines of an arbitrary degree, *in*, "Approximations with Special Emphasis on Spline Functions," (I. J. Schoenberg, Ed.), pp. 405–413, Academic Press, New York, 1969.
16. E. KAMKE, "Differentialgleichungen, Lösungsmethoden Und Lösungen," Becker & Erler Kom-Ges., Leipzig, 1943.
17. P. HARTMAN, "Ordinary Differential Equations," Wiley, New York, 1964.